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Marriage of Clifford Algebra and Finsler Geometry: A Lineage for Unification?

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The teleparallel Finsler connections on Riemannian metrics are more compatible with algebraic structure than the usual metric-compatible connections. Tangent and cotangent Clifford algebras come together with Finslerian teleparallelism to give rise to a Kałuża–Klein structure endowed with a canonical connection. The implications of this convergence for unification are explored.

1. INTRODUCTION

Riemannian geometry appeared to be incompatible with the algebraic structure of Euclidean geometry represented by the Euclidean group. In the early 1920s, Cartan showed the principal bundle way for incorporating Euclidean concepts in Riemannian geometry. The aim of this paper is to show that, provided that one assumes teleparallelism (TP), the frame bundle structure of Finslerian geometry, a generalization of Riemannian geometry, is more suitable than that of Riemannian geometry for the integration of algebraic structure into generalized geometry, with important implications for physics.

This paper complements recent work (Vargas and Torr, 2000) where we discussed the Kähler and Hestenes inclusions of Clifford algebras in the calculus and showed that the inner workings of the Kähler calculus prompt its own evolution into a calculus in a teleparallel Kałuża–Klein (KK) space. In that paper, we dealt only with the "Clifford algebra spouse" of the marriage referred to in the present title. In this paper, we are concerned with the "Finsler

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spouse," namely the evolution of the Finsler space into the aforementioned KK space.

In Section 4, we summarize TP, Finslerian teleparallelism (FTP), and an "enhanced" Kähler calculus for TP, as it is endowed with two Clifford algebras, both of them crucial. In Section 5, we compare our use of TP with other uses made of it. Since the enhanced Kähler calculus cannot be carried forward to FTP, we show the evolution of the FTP structure into a canonical KK space based on the same horizontal differential invariants (Section 6). This KK space admits the enhanced Kähler calculus. In Section 7, virtually canonical "super-Dirac" equations are shown, giving rise to a formidable geometric platform from which to launch field quantization, if so desired, In Section 8, the intimate relation between the gravitational sector of TP and quantum mechanics (QM) is discussed.

2. ON THE EVOLUTION OF FINSLER GEOMETRY

Finsler geometry receives its name from the doctoral thesis of Paul Finsler (1918), and deals with curves and surfaces in spaces endowed with metrics more general than the Riemannian ones. These metrics had been briefly mentioned by Riemann (1854) in his lecture for admission to the faculty at Göttingen, the same lecture where he introduced the metrics that go by his name. Although he eventually considered the issue of the curvature for Riemannian metrics (Riemann, 1861), he did not consider any similar problem for Finsler metrics. Even nowadays, Finsler geometry is often defined from the metric (but not affine) perspective of being the geometry of spaces where the length of curves is defined by

$$s = \int L\left(x^i, \frac{dx^i}{dt}\right) dt \tag{1}$$

A more general and modern perspective of Finsler geometry does not resort to curves for its definition. It rather involves affine structures (Vargas and Torr, 1993).

Let A denote the "Finslerian generalization" of "generalized affine geometry" B, both A and B comprising affine curvature and torsion. B in turn generalizes the "elementary affine geometry" or "geometry of the affine group," denoted C. In essence, the main component of B is the theory of affine connections. We shall refer to A as the theory of affine-Finsler connections. A, B, and C have metric-compatible specializations, denoted here A', B', and C'. In C', the metric is Euclidean, or equivalent to it by a coordinate transformation. In B' proper, the metric is Riemannian or, if Euclidean, the torsion of the connection is not zero. Similarly, in A', the metric may be Finslerian, but need not be so: if the metric is Riemannian, the Finslerian character of the geometry may come from the connection, i.e., from the rule for geometric equality of vectors at different points of the manifold, or parallel transport. A notable issue in A' is constituted by the so-called Cartan–Finsler or Cartan connection (Cartan, 1934). It is the canonical connection of distances of the general form given by Eq. (1). It is clear that, since one gets the Levi-Civita connection in the particular case that the distance is Riemannian, theoretical physics is short-changed by the limited concept of Finsler geometry embodied in the definition of this geometry given in the previous paragraph. We said "distance" because, unlike in Riemannian geometry, one must distinguish between the distance and the metric in Finsler geometry, as we shall see in the next section. We deal precisely with an option left out by such a restricted definition, namely teleparallel Finsler connections on Riemannian metrics.

The Cartan–Finsler connection, which constitutes the central topic of Cartan's monograph on Finsler spaces (Cartan, 1934), was treated without reference to general theories A and A'. Cartan did not return to this area of research, thus failing to develop a theory of A and A' that would parallel his work in Cartan (1923), where B' appeared as just a particular case of B. It is uncharacteristic that, in the monograph, Cartan used the tensor calculus, which is especially cumbersome in Finsler geometry, and that he did not resort to bundles. In the 1940s, Chern covered approximately the same ground as Cartan, but in a more elegant and traditional Cartan style, using differential forms and what we shall later refer to as the Finsler bundle (Chern, 1948). Although Chern introduced the appropriate soldering and connection forms, he did not explain what are the bases of the tangent bundle that are dual to his $n + n^2$ differential forms. Both eminent geometers thus failed to develop a formal theory of affine Finsler connections. In addition, as late as the 1990s, Chern still subscribed to the definition of Finsler geometry as in Eq, (1) (Bao and Chern, 1993).

Although several among the recent texts on the general area of Finsler geometry no longer present the foundations of this theory in terms of the length of curves, it is difficult to recognize in them the Finsler bundles. Also, their concepts of Finsler connections are often so involved that it is difficult to realize how those connections relate to the Lie groups and algebras of "flat affine and Euclidean geometry." A related review is provided in Section 2 of Vargas and Torr (1993), where references to the Finsler literature are given. We refrain from providing such references here since the proliferation of approaches can only confuse the reader and even challenge the expert. For instance, it took decades to realize that two famous connections, the Chern and Rund connections, are actually one and the same (Anastasiei, 1996).

Our development of a concept of affine Finsler connections involving bundles is mainly due to Clifton, as explained elsewhere (Vargas and Torr, 1993). It differs from other approaches in that it satisfies all of the following conditions: (a) Finsler connections are defined even in the absence of a metric function and/or metric, (b) the Finslerian character of the geometry arises from the connection, (c) the connection is viewed as a rule relating tangent vectors to, say, spacetime that have been moved to phase-spacetime (a manifold with coordinates t, x^i, u^j, where u^j denotes velocities), (d) when a metric is introduced, the emphasis of our work is on Finslerian connections on Riemannian metrics, which (e) allows us to avoid the thorny issue of metric compatibility with properly Finslerian metrics. Our development incorporates Cartan's view of metric geometry as a particular case of affine geometry. The general connections will be referred to as affine Finsler connections. If we further endow the manifold with a Riemannian metric, we restrict the bases to the usual frames and refer to metric Finsler connections and canonical metric Finsler geometry. In noncanonical metric Finsler geometry, the orthonormal bases are replaced with a subset of bases which are not orthonormal, but can be put in a one-to-one correspondence with them and assume their role (Vargas and Torr, 1996). Here, we shall be interested in the canonical case and shall not concern ourselves with the issue of whether existing experimental tests of isotropy (Haugan and Kaufmann, 1995) put constraints on the usefulness of Finsler metrics, which are anisotropic in a deep sense, in physics. Finsler metrics seem to arouse interest among cosmologists (S. Vacaru, private communication).

The use of Finsler metrics in avant-garde physical theories has been proposed by Vacaru (1997a, b). Apart from the fact that we do not use such metrics in this work, the unused richness present in the tangent bundle makes resorting to such untested physical theories an unnecessary complication and a distraction from our main course of action: since the mathematics that is involved in the geometric theory of Dirac equations (Vargas and Torr, 2000) and in FTP has a life of its own, we let it take us where it may.

3. THE FINSLER GEOMETRY OF MOVING FRAMES

The tangent vector at a point $m \in M$ of a differentiable manifold M is an equivalence class of curves $[x^\mu(\lambda), (a \leq \lambda \leq b)]$ through m which give the same values for $dx^\mu/d\lambda$ at $m \in M$. The set of numbers $dx^\mu/d\lambda$ at m change for any given curve under changes of coordinate system, but the changes are the same for all curves in the same equivalence class. These classes are therefore independent of coordinate system. Let TM be the tangent bundle of M, or set of all its tangent vectors. Let T_0M be the set of all the nonzero vectors in TM. In T_0M, we regard two vectors as equivalent if they belong to the same vector space, i,e., they are tangent at the same point $m \in M$ and they differ by a positive factor. The quotient of T_0M by this equivalence relation is called the sphere bundle *SM*. This is the manifold which, when *M* is endowed with a Lorentzian metric, it has the natural interpretation of geometric phase-spacetime. Its dimension is $2n - 1$, where n is the dimension of *M*. We proceed to introduce coordinates on *SM*. The sets (x^μ, $y^\mu = dx^\mu/d\lambda$) constitute a coordinate system on *TM*. Given the way in which *SM* was constructed from T_0M, we need only consider curves on *M* where the $dx^\mu/d\lambda$ and, therefore, the y^μ are not all zero. It is clear that the ratios $u^i = y^i/y^0$ constitute a coordinate system in regions where y^0 is not zero, the n values of the indices being $(0, i) = (0, 1, \ldots, n - 1)$. Among the curves on *SM*, there are those called natural liftings, such that the $u^i(\lambda)$ are not independent of the $x^\mu(\lambda)$, but we rather have $u^i = dx^i/dx^0$. *SM* is spanned by (dx^μ, du^i).

The set *BM* of tangent bases is usually fibrated over *M* with the linear group acting on the fibers. However, it may also be fibrated over *SM* via an isomorphism between tangent vectors to *M* and bases of "reduced" tangent vectors to *SM* (Vargas and Torr, 1993). "Reduced" stands for the fact that the extra $n - 1$ dimensions of the tangent vectors to *SM* have been made to disappear. The isomorphism gives rise to a refibration of *BM* as a frame bundle of "special" bases (of reduced tangent vectors). It will be denoted as *the Finsler bundle of bases*, with symbol $BM \to SM$. "Special" means that the first vector in the bases of the fiber over $s \in SM$ belongs to the equivalence class *s*. A more descriptive term could be "adapted." The fibers are standard, in the sense that the same group acts appropriately on all of them, namely the group of all transformations that leave the direction of the first basis vector unchanged. One could have chosen the second, third, . . . , basis vector to play this role, but only one. The group acting on the fibers becomes most familiar if we restrict the bases to the frames (orthonormal bases), in which case the Finsler bundle of bases becomes the *canonical Finsler bundle of frames*, denoted as $B'M \to SM$. For other Finsler bundles of frames, see Vargas and Torr (1996).

In the *canonical Finsler bundle of frames*, the group in the fibers is $O(n - 1)$ for properly Riemannian metrics. For Lorentzian signature, the group in the fibers also is $O(n - 1)$ if the direction chosen to perform the refibration is a time direction. Otherwise, the group would be $O(1, n - 2)$. In this case, one artificially puts together the time direction with an $(n - 2)$-dimensional spatial subspace, the remaining spatial subspace being used to create the Finslerian fibration. We shall of course ignore this unnatural possibility. The argument illustrates, however, that in the properly Riemannian case, one of the spatial directions is unnaturally singled out to play the role that the time direction naturally plays when the signature is Lorentzian. Hence "the canonical signature" of Finsler geometry is the Lorentzian signature, which raises the issue of whether the Minkowski spacetime should be considered as endowed with a pseudometric or as a Finsler space endowed with a particularly simple Finsler function.

The usual affine connections are specified by a set of $n + n^2$ independent differential 1-forms ω^μ and ω_ν^λ on the set of bases BM. They satisfy certain well-known properties and are such that $d\omega^\mu - \omega^\nu \wedge \omega_\nu^\mu$ and $d\omega_\lambda^\mu - \omega_\lambda^\nu \wedge \omega_\nu^\mu$ take the special form $d\omega^\mu - \omega^\nu \wedge \omega_\nu^\mu = R^\mu{}_{\nu\lambda}\, \omega^\nu \wedge \omega^\lambda$ and $d\omega_\lambda^\mu - \omega_\lambda^\nu \wedge \omega_\nu^\mu = R_\lambda{}^\mu{}_{\nu\pi}\omega^\nu \wedge \omega^\pi$. The most general 2-forms on the Finsler bundle of bases would be linear combinations of the $\omega^\nu \wedge \omega^\lambda$, $\omega^\pi \wedge \omega_\nu^\mu$, and $\omega_\lambda^\nu \wedge \omega_\pi^\mu$. In a similar vein, one can define an affine Finsler connection as follows.

A Finsler bundle of bases is said to be endowed with an affine-Finsler connection if it is endowed with $n + n^2$ linearly independent 1-forms (ω^μ, ω_ν^λ) such that:

(a) The ω^μ are the soldering forms, i.e., the result of applying all linear transformations which have nonvanishing determinant to the dx^μ.
(b) The ω_0^i are linear combinations of the dx^μ and du^i, and thus vanish on the fibers of $BM \to SM$.
(c) On the same fibers, the (pullbacks of the) forms ω_0^0, ω_0^i, and ω_j^k become the invariant forms of the group that leaves the direction of the first basis vector unchanged.
(d) The differential forms $d\omega^\mu - \omega^\nu \wedge \omega_\nu^\mu$ and $d\omega_\lambda^\mu - \omega_\lambda^\nu \wedge \omega_\nu^\mu$, respectively, called the torsion and affine curvature, are of the special form

$$d\omega^\mu - \omega^\nu \wedge \omega_\nu^\mu = R^\mu{}_{\nu\lambda}\omega^\nu \wedge \omega^\lambda + S^\mu{}_{\nu i}\omega^\nu \wedge \omega_0^i \tag{2}$$

$$d\omega_\lambda^\mu - \omega_\lambda^\nu \wedge \omega_\nu^\mu = R_\lambda{}^\mu{}_{\nu\pi}\omega^\nu \wedge \omega^\pi + S_\lambda{}^\mu{}_{\nu i}\omega^\nu \wedge \omega_0^i + T_\lambda{}^\mu{}_{ij}\omega_0^i \wedge \omega_0^j \tag{3}$$

One can show that terms of the T type (combinations of the $\omega_0^i \wedge \omega_0^j$) cannot appear in the torsion. Notice how postulates (a)–(c) define the Finsler bundle. The right-hand sides of Eqs. (2)–(3) are now more general than for the usual affine connections, but not as general as the most general 2-forms on the set of all bases. This degree of generality guarantees that the transport of vectors, though path dependent in general, is independent of cross section (Vargas and Torr, 1993).

Because the ω^0 and ω^i are the soldering forms, they are of the type

$$\omega^0 = A_0^0\, dt + A_j^0\, dx^j, \qquad \omega^i = A_0^i\, dt + A_j^i\, dx^j$$

$$L \equiv A_0^0 + A_j^0 u^j, \qquad A_j \equiv A_j^0, \qquad \sigma^j \equiv dx^j - u^j\, dt$$

$$\omega^0 = A_0^0\, dt + A_j^0 u^j\, dt + A_j^0(dx^j - u^j\, dt) = L\, dt + A_j \sigma^j \tag{4}$$

$$\omega^i = A^i_j\,(dx^j - u^j\,dt) = A^i_j \sigma^j \tag{5}$$

The curves on *SM* in which we are interested are the so-called natural liftings. They satisfy $\sigma^j = 0$, equivalently $\omega^j = 0$. Notice that ω^0 is the only ω^μ that is not zero on natural liftings. Since, *modulo* ω^j, we have

$$\int ds = \int \left[(\omega^0)^2 \pm \sum_j (\omega^j)^2 \right]^{1/2} = \int \omega^0 = \int L\,dt \tag{6}$$

it would seem that a concept of distance is defined even without a metric. However, $L\,dt$ changes from section to section, which is obvious if one considers the case when one multiplies the $\mathbf{e}_0$ vector by a factor. It is only when one has restricted the bundle in appropriate ways (like the restriction to the canonical Finsler bundle of frames) that $\int ds$ becomes independent of section. This is the way in which metric Finsler geometry becomes a specialization of affine Finsler geometry. The distance 1-form of Riemannian metrics is $L\,dt = (g_{00} + 2g_{0i}u^i + g_{ij}u^i u^j)^{1/2}\,dt$, where the g's are functions of the x coordinates only.

A most important feature of this view of Finsler geometry is that the horizontal forms are the ω^μ and ω^ν_0. This means that one can reconstruct the whole Finsler bundle (affine or metric) from a knowledge of these forms on a section of the same.

4. TELEPARALLELISM AND THE KÄHLER CALCULUS

In this section, we prepare the ground for a later reformulation of *Finslerian teleparallelism* (FTP) as a KK type theory endowed with a Kähler calculus based on two intertwined Clifford algebras. In Vargas and Torr (2000), we showed that the natural evolution of the standard Kähler calculus also leads one to the same KK space, endowed with a canonical teleparallel connection and without resort to Finsler geometry. The advantage of reaching the same KK structure through FTP, as in this paper, is that comparison with theoretical physics insinuates a canonical Dirac equation of great algebraic richness.

In Section 4.1, we present some immediate consequences for physical theory of a teleparallel spacetime connection of the usual, pre-Finslerian type. In Section 4.2, we discuss the consequences for physical theory of the original Kähler calculus, which is based on the Levi-Civita connection. In Section 4.3, we elaborate on increased opportunities for the physics if the Kähler calculus is developed with a teleparallel connection. In Section 4.4, we consider further enhancements for physics arising from the assumption that the connection of spacetime used in the Kähler calculus is not only teleparallel, but also Finslerian.

4.1. Consequences of Teleparallel Connections

In late 1915, Einstein created the theory of general relativity (GR). At that point, he could not have explained what *affine curvature* is in terms of transport of vectors, since the concept was introduced in mathematics only in 1917, by Levi-Civita. The curvature that Einstein considered was the *metric curvature*. TP is synonymous with the affine connection having zero affine curvature. It implies that there is an absolute or path-independent concept of equality of vectors from different tangent spaces. The following example is due to Cartan (1924a). The earth, punctured at the poles, is endowed with a teleparallel connection when we define as lines of constant direction the Rhumb lines, which include the meridians and the parallels. The unit vector in the North direction at one point of the earth is defined as being equal to the unit vector in the North direction at any other point. The same applies to other directions. The earth then has zero affine curvature. It, however, remains round, i.e., has non-Euclidean metric relations and nonzero metric curvature. Hence, because of historical reasons, 1915-GR could only have involved the metric curvature of spacetime. Present-day GR texts, however, introduce Riemannian spaces as already endowed with the Levi-Civita affine connection, even though. Einstein did not make any assumptions about the affine connection of spacetime.

The statement that the affine curvature of spacetime is zero can be written as an equation where the left-hand side is the metric curvature and the right-hand side also is completely geometric. TP is not incompatible in principle with 1915 (i.e., pre-affine) GR if one is able to identify this geometric right-hand side with the physical right-hand side of the standard Einstein equations. The geometric equations:

(a) Actually come *in a larger number than in GR* as they specify the full curvature and not just the Einstein tensor.
(b) Include *a greater variety of tensors* on the right-hand side of the subset obtained by their Einstein contraction, with the possibility, in principle, of accommodating in them the energy of fields of all types (Vargas and Torr, 1999b) and the gravitational energy in particular (Vargas and Torr, 1999a).
(c) Imply *localization of energy.*
(d) And come accompanied by *field equations* for the nongravitational sector of the theory, because the first Bianchi identity for TP, whether Finslerian or not, implies that the exterior covariant derivative of the torsion is zero

$$d\mathbf{\Omega} = \mathbf{0} \tag{7}$$

where $\mathbf{\Omega}$ is the torsion, and d is the exterior derivative when acting on a scalar-valued differential form and the exterior covariant derivative when acting on a tensor or Clifford-valued differential form, like the torsion.

4.2. Consequences of the Kähler Calculus for Levi-Civita Connections

The Kähler calculus is a calculus of tensor-valued differential forms that is to the Clifford product of differential forms what the Cartan calculus is to the exterior product of these forms (Kähler, 1960, 1962). It comprises a general theory of Dirac-type equations, which are of the form

$$\partial u = a \vee u \tag{8}$$

where ∂ denotes the sum of the exterior and interior covariant derivatives, a is a given tensor-valued Clifform, and u is the tensor-valued Clifform to be found. As formulated by Kähler (1961), the Dirac equation with EM coupling

$$i\hbar\gamma^{\mu}\,\partial_{\mu}\psi = \left(mc + \frac{e}{c}\,\gamma^{\mu}A_{\mu}\right)\psi \tag{9}$$

is the following simple particular case of Eq. (8):

$$-i\hbar\,\partial u = (im + eA)u \tag{10}$$

The solutions u are scalar-valued in principle, given the scalar-valuedness of $-m + ieA$ itself (there is no advantage in choosing any other valuedness). The original Dirac equation thus is just an entry-level equation. Indeed, a vector-valued or tensor-valued form would cause an increase in the richness of components of the solutions u so explosive as to be even unwanted (Vargas and Torr, 2000). Kähler considered tensor-valued differential forms (on spaces endowed with Levi-Civita connections). It is clear that, since the Clifford algebra is a quotient algebra of the tensor algebra, a calculus of Clifford-valued differential forms immediately follows from Kähler's work. This solves the problem of the "infinite explosion of valuedness" (Vargas and Torr, 2000). Physics has, therefore, barely explored the potential of spacetime Dirac equations, i.e., Kähler–Dirac equations for different a's, as opposed to gauge-Dirac equations. As Kähler himself pointed out (Kähler, 1962), his "calculus is for both, the quantum and relativistic theories, though its worth remains to be proven through its actual use by physicists." The issue now is the identification of the geometric object that generalizes the geometric potential.

The Kähler equation also comprises both fermionic and bosonic solutions (Vargas and Torr, 1998), the term fermionic being used here for solutions of Eq. (8) with nonnull a. The term bosonic solution is here used for what Kähler denotes as harmonic functions, which satisfy $\partial u = 0$ (Kähler, 1960).

Of particular interest is the property that the product of a fermionic solution for a given *a* and a bosonic solution is another fermionic solution for the same *a*.

4.3. Consequences of the Teleparallel Kähler Calculus

When postulating TP for unification, Einstein was hoping to get quantum mechanical effects through overdetermined systems of rather classical equations, admitting solutions only for discrete eigenvalues. With the different enhancements of the Kähler calculus, Dirac-type equations become increasingly geometric and one need not resort to this mechanism for quantization. Although Kähler developed his calculus for just the Levi-Civita connection, it is possible to extend his calculus to arbitrary (pre-Finslerian) connections, the teleparallel ones in particular (Vargas and Torr, 1998). Although the important property that the product of a fermionic and a bosonic solution is another fermionic solution for the same *a* no longer holds for arbitrary connections, this property is not lost since one recovers the Kähler calculus for the Levi-Civita connection as a weak torsion approximation. So there is altogether an increase in theoretical richness in this fact, apart from the benefit of being able to identify all the tangent spaces and therefore all the tangent algebras.

A general feature of the Kähler calculi is that they make the specification of the interior covariant derivative the natural complement to the specification of the exterior covariant derivative. Since TP provides the exterior covariant derivative of the torsion, Eq. (7), the completion of the determination of the torsion in TP is fulfilled through specification of its interior covariant derivative,

$$\delta\mathbf{\Omega} = \mathbf{J} \tag{11}$$

where **J** is, at this point, some current which one might try to describe phenomenologically (like the current in Maxwell's electrodynamics and the energy-momentum tensor in Einstein's equations). Equations (7) and (11) are to the vector-valued differential form torsion $\mathbf{\Omega}$ what Maxwell's equations are to the EM field *F*,

$$dF = 0, \qquad \delta F = j \tag{12}$$

In spite of these advantages of pre-Finslerian TP, there are two problems with it. One of them already shows at the classical level, namely the matching of the torsion with the EM field. As for the Dirac sector of the theory, the Kähler calculus has the following problem, regardless of connection. Consider the double Clifford product of the translation element, $d\mathbf{P} = \omega^{\mu}\,\mathbf{e}_{\mu}$, with itself:

$$d\mathbf{P}(\vee, \vee)\, d\mathbf{P} = dx^\mu \vee dx^\nu\, \mathbf{e}_\mu \vee \mathbf{e}_\nu = dx^\mu \wedge dx^\nu\, \mathbf{e}_\mu \wedge \mathbf{e}_\nu + 0 + 0 + g_{\mu\nu} g^{\mu\nu}$$
$$= dx^\mu \wedge dx^\nu\, \mathbf{e}_\mu \wedge \mathbf{e}_\nu + 4 \qquad (13)$$

The last term, 4, is a scalar-valued differential 0-form. In the same way as an *r*-form is evaluated (read integrated) on *r*-surfaces, one should assign in principle the value 4 to, possibly, each spacetime point. This peculiar feature has the potential to become a source of divergences in a physical theory based on this calculus. The solution is provided by Finslerian TP, or FTP. Let us first see what FTP has to offer to the classical sector of the physics.

4.4. Consequences of the Finslerian Teleparallel Kähler Calculus

The Finslerian fibration separates the Lorentz boosts from the rotations of *tangent* vectors (Vargas and Torr, 1999a). If, as Cartan argued (Cartan, 1924b), the EM form is represented by a differential 2-form in its role as a surface integrand (here denoted as a *cotangent* object) rather than by a *tangent* 2-tensor, the Lorentz group retains its relevance as a group of (passive) transformations, namely the subgroup associated with inertial frames of the infinite group of coordinate transformations. This decoupling of the boosts from the Finslerian fibers of *tangent* frames, where only the rotation group $O(3)$ acts, separates the temporal part from the spatial part of *tangent* vectors. Under this new group in the fibers, the Ω^0 component of the torsion behaves like a differential 2-form, i.e., like the EM field F, since the temparal component for *tangent* indices behaves like a scalar under the group $O(3)$.

The base space of the Finsler bundle of spacetime (i.e., the geometric phase-spacetime) has a remarkable property, namely that the subluminal 4-velocities constitute a canonical field over it (Vargas and Torr, 1999b). In the “acceleration vector-valued 1-form” that corresponds to this “velocity vector-valued 0-form,” one is able to identify the Lorentz force and to tentatively identify different pieces of the torsion with different interactions. This also explains the extremely short range of the weak interactions (which have zero classical range), which in turn allows for the possibility of a nonconstant cosmological “constant,” etc.

Once an interpretation has been provided for the different pieces of the torsion and, in particular, for the EM field, one can start to extract consequences from the geometric Einstein equations referred to above (Vargas and Torr, 1999a). One such consequence is that nonhomogeneous electric and magnetic fields act as sources of gravitational fields (this is in addition to the extremely negligible deformation of the spacetime structure through the quadratic terms in the energy-momentum tensor of these fields). These effects may be in principle the cause of the discrepant measurements of Newton’s constant G (Vargas and Torr, 1999a).

If the torsion generalizes the EM field, then the potential for the torsion, namely the translational line element, generalizes the EM potential. One should use this geometric potential as the factor *a* in the Kähler equation, thus obtaining a controlled explosion of valuedness, i.e., confined to the finite dimensionality of the Clifford algebra. The problem with this structure, however, is that, whereas the tangent frames that we use in the Finsler bundle are 4-dimensional, the base space of the bundle is a 7-dimensional manifold (recall that the tangent Finsler bundle is constituted by bases of *reduced* tangent vectors). Related to this, TP no longer provides constant sections in the Finsler bundle, not even in the Finsler bundle of Minkowski spacetime (the sections are too large, so to speak). It is at this point that the reformulation of FTP as a KK theory enters the picture. But, first, let us deal with the potential interference in the mind of readers between this version of TP and the alternative version(s) of it in the literature.

5. COMPARISON OF APPROACHES TO TELEPARALLELISM

Our approach to teleparallelism (TP), which we have shown (Vargas and Torr, 1999a) to be closely related to Einstein's (1930a, b) approach, has been overlooked in the literature, in spite of its naturalness. It is natural because the teleparallel version of the geometric equations known as the second equation of structure and the first Bianchi identity become field equations, namely generalizations of the Einstein and first pair of Maxwell's equations. The theory which thus emerges is completely inimical to the teleparallel theories comprised in the general framework of what is known as metric-affine theory of gravity (Hehl *et al.*, 1995). The reason behind their opposing characters is that the marriage of Clifford algebra and Finsler geometry does not give rise just to a gravitational theory, but rather to a full classical sector and a virtually canonical geometric Dirac sector. The emergence of so different theories from the same postulate has to do with the fact that the teleparallel alternatives to our proposal do not contain the structural equations of TP as field equations, but rather derive these through ad hoc variations that use reasonable, but ad hoc actions: these involve the curvature as a constraint contributing a Lagrange multiplier term to the action, or they involve only the Ricci contraction, again as a constraint, or do not involve the affine curvature at all, and even then there are several options (Müller-Hoissen and Nitsch, 1985). This difference in character would exist even if our use of TP concerned only the usual, not-yet-Finslerian theory of affine connections, were it not for the fact that the identification of geometric quantities with physical fields is possible only in Finsler bundles. Thus, nothing of what these theories have to say has any bearing whatsoever upon the statements and equations of our proposal, and the statements of the paragraph that contains Eq. (7) in particular.

In spite of these considerations, it is worth dealing with specific issues that arise in those alternatives, since they illustrate the profound implications of our proposal. In spite of the existence of many versions of teleparallel metric-affine gravity, there seems to be agreement among practitioners that, to different degrees depending on the specific theory, difficulties arise which are known as the problem with the Cauchy formulation. It consists in nonpredictable behavior of the torsion (Kopczynski, 1982) as well as other problems caused by the attempt to remove this unpredictability (Gönner and Müller-Hoissen, 1984). The Cauchy problem of teleparallel metric theories of gravity is in essence the same as in GR, in consonance with the belief that (with blatant disregard for the different implications of the Levi-Civita and teleparallel connections for the problem of equality of vectors and localization of energy in GR) some teleparallel theories in Riemann–Cartan spacetime are viewed as empirically indistinguishable from GR (Mielke, 1992) and classically equivalent to it (Mielke, 1999). On the other hand, the Cauchy problem of the present proposal is the trivial one which corresponds to the specification of the full curvature, which is amply discussed in Section 7 for its quantum mechanical implications. As pointed out by Cartan (1922), “Einstein’s gravitational equations amount to only ten linear combinations of Riemann’s 20 symbols. . . It is quite disconcerting that only these quantities have been considered by physicists.” It is high time to consider the implications of the fact that TP implies 20 curvature equations rather than just 10.

Equation (8), which is instrumental in obtaining the torsion, is our sole field equation which is not literally an equation of structure or Bianchi identity. Notice, however, that the first equation of structure specifies the torsion and that the first Bianchi identity (FBI) specifies its exterior covariant derivative. Since we postulate the FBI as a field equation, the specification of the torsion (in the sense of specifying its total derivative in the sense of Kähler) is completed by the specification of the interior covariant derivative (also in the sense of Kähler). This connection-dependent derivative is obtained within the tenets of the Kähler calculus, and its mathematical suitability is unassailable by any theory which does not resort to this calculus (note that the interior covariant derivative of the Kähler calculus does not coincide in general with the one obtained through the Hodge dual, the Levi-Civita connection being an exception). This theory thus transcends in character anything offered by metric-affine theory (*of gravity*, that is, as its authors intended it).

The widely held belief that the torsion cannot generalize the EM field because of the mismatch between the respective symmetries [translational in the case of the torsion and U(1) in the case of the EM field] ignores the richness and subtleties of Finsler geometry, as shown in later sections. The correct perception of the illegitimate character of such identification in the usual or pre-Finslerian context (for instance, a boost would mix different parts of the torsion and would amount to mixing EM and weak fields in a clearly unphysical manner) together with the lack of in-depth knowledge of Finsler geometry may have prompted physicists to look for the less natural interpretations of TP that pervade the literature.

6. KALUŻA–KLEIN REFORMULATION OF FINSLERIAN TELEPARALLELISM

The bundle approach to Finsler geometry that we motivated in Section 2 and outlined in Section 3 alerts us to the fact that the basic differential invariants defining the Finsler geometry are the ω^{μ} and the ω_0^i. Together with the $O(3)$ group, they contain the essence of the geometry since one can reconstruct from them the whole Finsler bundle. These differential invariants define the differential translation $d\mathbf{P}$ and the "acceleration 1-form", $d\mathbf{u}$ ($= d\mathbf{e}_0 = \omega_0^i\,\mathbf{e}_i$). Since the form that is dual to $\mathbf{u}$ is ds, the following construction suggests itself immediately.

Let M^4 be a 4-manifold endowed with a pseudo-Riemannian metric of Lorentzian signature and a compatible teleparallel connection. The tangent spaces at different points on M^4 can be identified so as to constitute just one vector space V^4. Let $\mathbf{a}_{\mu}$ be a constant (pseudo)-orthonormal basis of V^4 and let ω^{μ} be the basis of differential 1-forms dual to $\mathbf{a}_{\mu}$. Let $\mathbf{u}$ be a unit tangent vector in the tangent space V^1 to a differentiable manifold M^1. Let s be the coordinate on M^1 dual to the unit tangent vector $\mathbf{u}$. On $V^4 \otimes V^1$, let $\mathbf{u}$ be the fifth element $\mathbf{a}_4$ of bases $(\mathbf{a}_A)$, with the indices $A, B, \ldots$ running from 0 to 4. Needless to say, $\omega^4 = ds$. The only unknowns in this metric are the $g_{4\mu}$, i.e., we have

$$g_{AB} = \mathbf{a}_A \cdot \mathbf{a}_B = \begin{pmatrix} 1 & 0 & 0 & 0 & g_{40} \\ 0 & -1 & 0 & 0 & g_{41} \\ 0 & 0 & -1 & 0 & g_{42} \\ 0 & 0 & 0 & -1 & g_{43} \\ g_{40} & g_{41} & g_{42} & g_{43} & -1 \end{pmatrix} \tag{14}$$

There is also the spacetime metric, which is buried in the soldering forms ω^{μ}. It would become part of g_{AB} if we did use tangent vectors dual to coordinate bases of 1-forms. We do not, since coordinate tangent bases are not constant and their use complicates the form of the canonical connection that we are about to obtain.

A translation element $d\mathfrak{p}$ is defined on $M^4 \oplus M^1$ by $d\mathfrak{p} = \omega^{\mu}\mathbf{a}_{\mu} + ds\mathbf{u}$. One requires

$$\omega^A \cdot \omega^B = 0 \qquad \text{for} \quad A \neq B \tag{15}$$

For $A = B$, we leave the product indicated, $\omega^A \cdot \omega^A$. These are functions whose interpretation will become clear in the following.

It is clear that one can obtain a (pseudo)-orthonormal canonical tangent basis with signature $(1, -1, -1, -1, -1)$. The basis of 1-forms dual to the $(\mathbf{e}_A)$ is readily obtained by means of

$$d\mathfrak{p} = \mathbf{a}_A \omega^A = \mathbf{e}_A \omega'^A \tag{16}$$

The result is

$$\omega'^0 = \omega^0 + g_{40}\, ds, \qquad \omega'^i = \omega^i + g_{4i}\, ds, \qquad \omega'^4 = G^{1/2}\, ds \tag{17}$$

$$G \equiv 1 + (g_{40})^2 - (g_{41})^2 - (g_{42})^2 - (g_{43})^2 \tag{18}$$

It is important to note that, whereas $\omega^A \cdot \omega^B$ is zero for $A \neq B$, the same is not the case for $\omega'^A \cdot \omega'^B$. Also, with the canonical basis, the interpretation of the unit vector in the "fifth dimension" as the 4-velocity is lost.

To obtain the metric tensor without tensor products other than the exterior one, we postulate a null double dot product of $d\mathfrak{p}$ with itself:

$$d\mathfrak{p}(\cdot, \cdot)\, d\mathfrak{p} = \omega^A \cdot \omega^B \mathbf{a}_A \cdot \mathbf{a}_B = \omega^0 \cdot \omega^0 - \omega^1 \cdot \omega^1 - \ldots - ds \cdot ds \tag{19}$$

so that the equation

$$ds \cdot ds = \omega^0 \cdot \omega^0 - \omega^1 \cdot \omega^1 - \omega^2 \cdot \omega^2 - \omega^3 \cdot \omega^3 \tag{20}$$

becomes an alternative form of the metric using only products pertaining to Clifford algebra. Furthermore, instead of Eq. (13), we now have

$$d\mathfrak{p}(\vee, \vee)\, d\mathfrak{p} = d\mathfrak{p}(\wedge, \wedge)\, d\mathfrak{p} \tag{21}$$

Note the disappearance of the unwanted term 4.

One obtains a canonical connection for spacetime TP by setting, instead of null torsion, the spacetime part of the connection to zero,

$$\omega_\alpha^\beta = 0 \tag{22}$$

(i.e., spacetime teleparallelism) and metric compatibility:

$$\omega_{AB} + \omega_{BA} = dg_{AB} \tag{23}$$

where $\omega_{AB} \equiv \omega_A^C g_{CB}$. The only nonnull components of the connection are

$$\omega_{4\rho} = dg_{4\rho}, \qquad \omega_4^\rho = \eta_\rho(\omega_{4\rho} - g_{4\rho}\omega_4^4), \qquad \omega_4^4 = G^{-1}\eta_\rho g_{4\rho}\, dg_{4\rho} \tag{24}$$

The connection components with mixed indices satisfy the relationship

$$\omega_4^4 = \omega_4^\rho g_{4\rho} \tag{25}$$

For details regarding the obtaining of these results see Vargas and Torr

(1997a). It is interesting to notice that we assumed $\omega_\alpha^\beta = 0$ and, using metric compatibility, obtained $\omega_\alpha^B = 0$, equivalently $d\mathbf{e}_\alpha = 0$.

With this connection, with the aforementioned dot products in both algebras and with the standard rule of the Kähler calculus to obtain interior covariant derivatives, one gets (Vargas and Torr, 2000), for a 1-form $\alpha = a_\mu \omega^\mu$,

$$\delta\alpha = a_{0;0}\omega^0 \cdot \omega^0 + a_{1;1}\omega^1 \cdot \omega^1 + a_{2;2}\omega^2 \cdot \omega^2 + a_{3;3}\omega^3 \cdot \omega^3 \qquad (26)$$

This is a major result that we wanted to reach to make a point. The interior covariant derivative and therefore the Kähler derivative of a 1-form has been determined. Instead of the traditional $a_{\mu;}{}^{\mu}$, we have a function to be integrated on the trajectories of particles. Hence the disappearance of the constant term in Eq. (13) is accompanied by this new feature, which we expect to have an impact on further developments of the theory.

To summarize, the double Clifford algebra is defined by the fields $g_{4\mu}$ and the functions $\omega^0 \cdot \omega^0$, $\omega^1 \cdot \omega^1$, $\omega^2 \cdot \omega^2$, and $\omega^3 \cdot \omega^3$ of spacetime curves. These are in turn given by the $g'_{\mu\nu}$, defined by $d\mathbf{P} \cdot d\mathbf{P} = g'_{\mu\nu}\, dx^\mu\, dx^\nu$. Our connection is the canonical connection of $M^4 \oplus M^1$. Finslerian TP thus has a well-defined and canonical Kähler derivative. Hence, there is a canonical left-hand side of the Eq. (8). The issue then is whether there also is a canonical right-hand side, i.e., a canonical Clifford-valued Clifform a.

7. (QUASI-)CANONICAL KÄHLER–DIRAC EQUATIONS

We have shown the potential of FTP to produce a superseding theory of EM and GR (Vargas and Torr, 1999a) and to encompass in the torsion all the nongravitational fields (Vargas and Torr, 1999b). In this section, we show that the differential invariants that define the teleparallel Finsler space *almost* determine a canonical Kähler–Dirac equation and that the gauge transformations may after all be spacetime symmetries. A better understanding of this novel proposal might conceivably yield a unique Kähler–Dirac equation. In the next section, we show how Finslerian TP brings together GR and the Dirac sector.

The left-hand side of the hypothesized quasicanonical Kähler–Dirac equation has been determined, since the action of the operator ∂ is well defined by the canonical connection of the teleparallel KK space. The issue then is to find a canonical a for Eq. (8), or at least one which may be determined on canonical mathematical grounds except for, perhaps, minor help from physics. The a must depend only on the invariants that define the structure and should not be unlike the potential that goes into the gauge-covariant derivative. The torsion's potential function is the vector-valued 1-form $d\mathbf{P}$. Hence one may expect that the "canonical Kähler–Dirac equation" will resemble something like

$$\partial\psi = d\mathbf{P} \vee \psi \tag{27}$$

As we later show, two different Kähler–Dirac equations are contained in Eq. (27), depending on how $d\mathbf{P}$ is understood. There is an additional, simple possibility. The reformulation of Finsler geometry as a KK geometry produces the option that $d\mathfrak{p}$ rather than $d\mathbf{P}$ itself is the factor a that we are looking for, so that the equation

$$\partial\psi = d\mathfrak{p} \vee \psi \tag{28}$$

results. But then, where does the mass term hide? More importantly, how could a theory such as this one, based on classical differential geometry, account for the unit imaginary and $\hbar$ and for the probabilistic nature of QM?

The issue of the unit imaginary is relatively simple. As Hestenes (1966) has shown, the unit pseudoscalar, to name just one example, can play the role of the unit imaginary. In our case, a different unit imaginary may be already implicit in Eq. (27), depending on how we view $d\mathbf{P}$. To make the point clear, consider the familiar case of the representation of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in Clifford algebra. We may think of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as the spatial projection in spacetime of $x^\mu\ \mathbf{e}_\mu$ or as the radius vector in 3-space. Let σ_i represent a canonical basis in 3-space and let γ_μ represent a canonical basis in spacetime. The spatial projection will be given as $x^i\gamma_i$ and the radius vector in 3-space as $x^i\sigma_i$, which can be rewritten as $x^i\gamma_i \vee \gamma_0$. In our case, we may view $d\mathbf{P}$ as a projection on spacetime of $d\mathfrak{p}$ or as the translation element of spacetime. In the first case, $d\mathbf{P}$ is given by $\omega^\mu\gamma_\mu$. In the second case, it is given by ω'^μ $\mathbf{e}_\mu \vee \mathbf{e}_4$. Expressing these quantities in terms of the ω'^μ and $\mathbf{a}_\mu$s, the vector $\mathbf{u}$, whose square is minus one, emerges as a factor and thus may play the role of the unit imaginary. Full details cannot be addressed here. We hope to have shown, however, that there are natural options for geometric unit imaginaries to appear in Kähler–Dirac equations and that perhaps one of these units is already contained in Eq. (27), depending on which $d\mathbf{P}$ we have in mind when we write this equation.

The issue of the mass term, although also conjectural at this point, could be understood in terms of a reformulation of the canonical Kähler–Dirac equation. It refers to pure Kähler–Dirac fields, unlike the dualistic Dirac equations of the physics, which deals with particles in fields. That equation is constituted by a collection of component equations, each of which couples forms of different ranks (through the exterior and interior derivatives) and of different valuedness (through exterior and interior multiplication by the vector-valued translational element). Each of these component equations represents the equality of two integrands, and thus the equality of their integrations. There will be physical problems, or approximations to physical problems, where one may manipulate these integrands, separating the “test particle" from the particles that are described in the form of the external field, thus giving rise to a dualistic description and its associated wave function. Such a course of action may in principle give rise to the mass term on the one hand and to the charge factor in the external field term on the other.

The issue of $\hbar$ is far more subtle and requires a lengthy explanation. It is postponed to the next section. We proceed to discuss gauge transformations. These transformations here emerge in the tangent bundle of the KK space, rather than in some ad hoc internal space. The group $SU(2)$ of microphysics is already latent in the fibers of the Finsler bundle of macrophysics in the form of the group $O(3)$ of its fibers (Vargas and Torr, 1999a). We shall now discus $U(1)$. It is well known that rotations, whether regular or hyperbolic (boosts), are given by exponentials of bivectors in Clifford algebras. Translations, on the other hand, are conspicuously absent from these algebras. The Finsler bundle of frames introduces two major modifications in the relation of the transformations in the Euclidean–Poincaré groups and the algebras themselves. First, boosts have disappeared from the groups in the fibers of Finslerian fibrations. They have not, however, disappeared altogether, since they are still present in the horizontal forms of the base space, namely in the connection forms ω_0^i. In the reformulation of Finsler geometry as a KK theory, the boosts appear in the following way. Rewrite the equation for the metric as

$$\omega^0 \cdot \omega^0 - ds \cdot ds = \omega^1 \cdot \omega^1 + \omega^2 \cdot \omega^2 + \omega^3 \cdot \omega^3 \tag{29}$$

It is clear that the ordinary rotations alter the individual terms on the right-hand side without altering their sum. Equivalently, a 1-dimensional boost in (t, s) subspace does not alter the left-hand side of Eq. (29). Associated with a 3-dimensional interval given by a specific evaluation of the 1-forms $(\omega^1, \omega^2, \omega^3)$, we can change ds and ω^0 in such a way that $\omega^0 \cdot \omega^0 - ds \cdot ds$ does not change. Thus, the second major modification is that a translation between two spacetime points separated by some (specific evaluation of the linear forms) ω^i is now represented by a boost in the cotangent subspace spanned by (ω^0, ds). The translation uses up a time ω^0, equivalently a proper time ds, the two being related as in a relativistic boost. So, translations also have to do after all with Clifford algebras.

According to this picture, the group $SO(1,1)$ is the group of boosts in the two-dimensional subspace of KK spanned by ds and ω^0. Locally, $SO(1,1)$ is isomorphic to $SO(2)$, whose spinorial equivalent is $U(1)$. Hence the $U(1)$ group of electrodynamics is actually a group of (phase)-spacetime symmetries. In other words, it is classical differential geometry as these transformations are directly related to the tangent–cotangent bundle complex, rather than to any internal symmetry. The group $SU(2)$ has a similar origin, namely related to the subgroup of rotations of the tangent Lorentz group. In the Finslerian refibration of the set of inertial frames, the rotations and boosts part ways to some extent. In the KK refibration of the Finsler structure, the boosts disappear as such, but reemerge as boosts in just one spatial dimension. A similar view of electroweak theory as spacetime theory has been independently developed by Pandres (1998, 1999). Although he does not state it explicitly in his papers, Pandres agrees that he is doing TP (private communication). He does not think that his theory needs the Finslerian setting to be justified. We claim that, when the Pandres theory is viewed as pertaining to this setting, one cancels the potential criticism that EM fields in one frame become electroweak mixtures in a boosted frame. Finally, the issue arises of where $SU(3)$ lies. Elaborating on pioneering work by Chisholm and Farwell (1992), it has been shown that the complexity of the internal structure of the strong interactions can be based on orientation groups in Minkowski space geometry (Schmeikal, 1996). This work carries over to the tangent vector space of teleparallel manifolds through the identification of all tangent vector spaces. In our development of FTP, we have not yet made contact with $SU(3)$ because we have barely started to study the Clifford structure of our formalism.

Note that we have obtained the explanation of why the phase factor rather than the phase matters in the standard version of electrodynamics (Wu and Yang, 1975) (recall that the phase is a multivalued function of the phase factor). The lack of physical meaning of the information that is contained in the phase, but not contained in the phase factor, has to do with the fact that we are using as symmetry group $e^{i\varphi}$ rather than e^{φ} (to be precise, $e^{\alpha}\varphi$, as explained in the next paragraph). This last factor contains the same information as φ since the function $f = arg(e^{\varphi})$ is single-valued, unlike $f = arg(e^{i\varphi})$. Let us now give an indication of why nonunitarity is not a problem.

To make matters simple, we have compared $e^{i\varphi}$ with e^{φ}. In reality, we should have compared $e^{i\varphi}$ with $e^{\alpha\varphi}$, where α is a tangent vector whose square is $+1$ (tangent vector here means an element of the direct sum of the tangent vectors to M^4 and M^{-1}). It represents a boost in a 2-dimensional subspace that is a remnant of the ordinary form of these boosts. The probability density will not be conserved because it indeed should-not be conserved under boosts.

8. ON THE RELATION OF GRAVITATION TO QUANTUM MECHANICS

A completely geometric Kähler–Dirac equation gives rise to a completely geometric current, which specifies the interior covariant derivative of the torsion through Eq. (   ) and achieves the geometric closure of the system of field equations. The depth of the relation between GR and QM goes beyond the coexistence of the Kähler–Dirac and gravitational field equations in this system. It has to do with the nature of its initial value problem and, specifically, with the fact that the full curvature and not just the Einstein tensor is given. Let us start by considering the initial value problem in GR.

In GR, one postulates the Einstein tensor and integrates the field equations to obtain the metric and thus the *full metric curvature*, i.e., the Einstein *and* the Weyl tensors. It seems as if one gets out of the field equations more (the Weyl tensor) than one puts in. The additional information on the curvature is contained in the initial conditions for the Cauchy problem of GR. It is then clear that, if the field equations specify not only the Einstein tensor, but also the full tensor, one is going to have a completely different, actually trivial, Cauchy problem, as we now describe. First, one should observe that the mere fact that the solving of the Cauchy problem of GR requires such initial conditions does not bode well for the problem of quantization of GR, after first assuming that gravity has to be quantized. What does a graviton care about initial conditions on a hypersurface?

To make matters simple, we proceed with the problem of initial conditions in three steps. When the affine curvature is zero, the system of equations.

$$d\omega_\lambda^\mu - \omega_\lambda^\nu \wedge \omega_\nu^\mu = 0 \tag{30}$$

on the bundle of bases of affine space is integrable. In addition, the system $d\mathbf{e}_\mu = \omega_\mu^\nu \mathbf{e}_\nu$, itself is integrable. Through integration, it yields the linear group, namely $\mathbf{e}_\mu = A_\mu^\nu \mathbf{a}_\nu$ in terms of some fixed basis $(\mathbf{a}_\nu)$. The n^2 integration constants, A_μ^ν are the entries of the most general nonsingular $n \times n$ matrix. This process, however, does not yield a formula for comparing a basis $(\mathbf{a}_\mu)$ at a point of coordinates (x^μ) with a basis $(\mathbf{b}_\mu)$ at an arbitrary point $(\mathbf{x}'^\mu)$, which is the actual situation that one faces when there is not integrability. For this, we have to integrate $d\mathbf{e}_\mu = \omega_\mu^\nu \mathbf{e}_\nu$ along a line, with $(\mathbf{a}_\mu)$ as initial condition. Any line will do, since TP makes this integration path independent.

We go one step further, namely to the case when we have zero torsion and zero affine curvature (i.e., in locally affine spaces like planes, cones, and cylinders). The integration of the system of equations $d\mathbf{P} = \omega^\nu \mathbf{e}_\nu$ and $d\mathbf{e}_\mu = \omega_\mu^\nu \mathbf{e}_\nu$ between points (x^μ) and (x'^μ) is independent of path and gives us the pair $(\mathbf{P}, e_\mu)$ attached to (x'^μ) in terms of the initial conditions $(\mathbf{Q}, \mathbf{a}_\mu)$ at (x^μ).

In the third step, we consider nontrivial TP, i.e., nonzero torsion and zero affine curvature. As in locally affine space, one still has to integrate for the frame $(\mathbf{P}, \mathbf{e}_\mu)$, after integrating for both the translational and linear part of the connection. The result for the integration of the translation now is path dependent and plays the role of the integration of $A_\mu \, dx^\mu$, since $d\mathbf{P}$ generalizes the EM potential. As for the obtaining of the metric itself, it does not alter in a significant way the process just described. The 4-dimensional metric is simply the product $d\mathbf{P} \cdot d\mathbf{P}$.

We present an example of a sophisticated system of nonlinear field equations with initial conditions at a point. Assume the Minkowski metric. Define the tensor $\beta = \mathrm{I}^{\lambda}_{\mu\rho}\mathbf{e}^{\mu} \otimes \mathbf{e}_{\lambda} \otimes \mathbf{e}^{\rho}$. If we set $d\beta = 0$, we get

$$d\Gamma^{\nu}{}_{\nu\lambda} + (\Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\nu}{}_{\sigma\rho} - \Gamma^{\nu}{}_{\phi\lambda}\Gamma^{\alpha}{}_{\mu\rho} - \Gamma^{\nu}{}_{\mu\sigma}\Gamma^{\alpha}{}_{\lambda\rho})\, dx^{\rho} = 0 \tag{31}$$

These are the Muraskin equations, which we name after Murray Muraskin for the very large number of computer studies that he performed on them (Muraskin, 1995). As discussed elsewhere (Vargas and Torr, 1997b), this system of equations is very similar to typical systems of equations satisfied by the torsion, except that, since we are now dealing with a tensor-valued 0-form, we have total differentials instead of exterior and interior covariant derivatives (read differentials). It has as initial conditions the $\Gamma^{\mu}{}_{\mu\lambda}$ at a point. One can chose initial values such that solitonic solutions arise. In Muraskin's interpretation, these solutions might represent (a) bounded, multi-wavepacket solutions without uncontrollable spreading, (b) vacuum, the region between the packets, constituted by very close small oscillations with a band structure which (c) evolves spontaneously into the packets, back into the vacuum, back into the packets, etc. The field equations of the vacuum are the same as in the spacetime occupied by the packets themselves. The difference between particles and vacuum lies simply in the particular form that the solution takes in a given piece of spacetime, i.e., packet or background. The picture that this implies for the system of equations emerging for TP from the marriage of Clifford algebra and Finsler geometry is that of (a) bosonic solutions (the vacuum plus its packets) for $\mathbf{J} = 0$ and (b) fermionic solutions where the macroscopic and microscopic solutions couple through $\mathbf{J}$.

The above applies only if there is only one source of torsion. Suppose that we had many source points of torsion, at least as many as there are sources of EM radiation in the universe that contribute to the background (including thermal and zero point) at any particular spacetime point. We would have to switch initial conditions in each one of them, rather than just at one point. These individual sources would give rise to their own, modulated backgrounds, which would interfere with each other in a nonlinear (the system that the solutions obey is not linear) and completely nontraceable way. A stochastic torsion background seems to be the natural outcome of this situation. The connection of spacetime is therefore stochastic and effectively nondeterministic, given the unfathomable background of which the bosonic (soliton) solutions make part and in which the fermionic solutions live. Thus, if the picture that has emerged in this paper works, the representation in the Kähler–Dirac equation of the stochastic process that emerges from the boundary conditions for teleparallelism would be in the $\hbar$, since it is the nonzero value of $\hbar$ which is at the root of the nondeterministic character of QM. Incidentally, the nonlinear nature of the field equations satisfied by the vacuum implies that a given background may look very different in the proximity and far away from matter, the differences being just a matter of scale in the linear case.

It remains to be seen what effects this stochasticity has for gravitation. Consider, for instance, the region around a neutral mass at distances large by comparison with the size of the mass. In the absence of any (stochastic) background, the torsion field would clearly be zero at those distances and so would the affine curvature, by hypothesis. The connection would therefore be zero and we would have flat spacetime. Newton's second law would not work. But it does. It is the stochastic torsion background (read the vacuum fluctuations) that invalidaes the argument for flat spacetime. According to the picture that emerges here, Sakharov's conception of gravitation as an effect of vacuum quantum fluctuations (Sakharov, 1968) is not just a clever idea for a theory of gravitation alternative to GR. It is actually consistent with the theory of gravitation contained in FTP. The $\hbar$ of quantum physics is intricately tied to fluctuations of the vacuum, which in turn is intricately tied to gravitation. A zero-torsion field would not be found in vacuum. If at all, it would be found in superconductors, away from its vortices.

9. CONCLUDING REMARKS

We have shown that Finslerian TP has a life of its own. When one lets this life freely develop, as an alternative to trying to make it fit preconceived physicists' practices, a formidable theoretical framework emerges. Einstein (1930a, b) tried TP as a fundamental postulate for one of his attempts at unification. He expected that, if this or any other of his attempts had been successful, QM would have become unnecessary. His TP was not Finslerian. However, he might have recognized the need for it be so, had the Finsler geometry and Clifford calculus of his time been more advanced. The irony is that a geometric form of QM, embodied in a Dirac-type (i.e, Kähler's) equation, is an integral part of FTP, since it can be derived, canonically or quasicanonically, from the horizontal differential invariants that define FTP. In this way, the closed system of teleparallel field equations becomes fully geometric, as he dreamt. Einstein had the right intuition in postulating TP. With insufficient mathematical tools, he failed to guess the final form of the script.

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REFERENCES

Anastasiei, Mihai (1996). A historical remark on the connections of Chern and Rund, *Contemporary Mathematics*, **196**, 171–176.

Bao, David, and Shiing-Shen Chern (1993). On a notable connection in Finsler geometry, *Houston Journal of Mathematics*, **17**(1), 135–180.

Cartan, Élie (1922). Sur les équations de la gravitation d'Einstein, *Journal de mathematiques pures et appliquées*, **1**, 141–203.

Cartan, Élie (1923). Sur les variétés à connexion affine et la théorie de la relativité généralisée, *Annales de l'École Normale*, **40**, 325–412.

Cartan, Élie (1924a). Les récentes généralizations de la notion d'espace, *Bulletin des Sciences Mathématiques*, **48**, 294–320.

Cartan, Élie (1924b). Sur les variétés á connexion affine et la théorie de la relativité généralisée, *Annales de l'École Normale*, **41**, 1–25.

Cartan, Élie (1934). *Exposés de géometrie*, Hermann, Paris.

Chern, Shiing-Shen (1948). Local equivalence and euclidean connections in Finsler spaces, *Science Reports Tsing Hua University A*, **5**, 95–121.

Chisholm, J. S. R., and R. S. Farwell (1992). Tetrahedral structure of idempotents of the Clifford algebra $C_{3,1}$, in: Artibano Micali, Roger Boudet, and J. Helmstetter, Editors, *Clifford Algebras and Their Applications in Mathematical Physics*, Kluwer, Dordrecht, pp. 27–32.

Einstein, Albert (1930a). Théorie unitaire du champ physique, *Annales de l'Institut Henri Poincaré*, **1**(1), 1–24.

Einstein, Albert (1930b). Auf die Riemann-Metrik und Fern-Parallelismus gegründete einheitliche Feldtheorie, *Mathematische Annalen*, **102**, 685–697.

Finsler, Paul (1918). *Über Curven und Flächen in Allgemeinem Räumen*, Dissertation, Göttingen. Reprinted (1951) Birkhäuser, Basel.

Gönner, H., and Fockert Müller-Hoissen (1984). Spatially homogeneous and isotropic spaces in theories of gravitation with torsion, *Classical and Quantum Gravity*, **1**, 651–672.

Haugan, Mark P., and Thierry F. Kauffmann (1995). *Physical Review D*, **6**, 3168–3175.

Held Friedrich W., J. Dermott McCrea, Eckehard W. Mielke, and Yuval Ne'eman (1995). Metric-affine gauge theory of gravity: Field equations, Nöther identities, world spinors, and breaking of dilation invariance, *Physics Reports*, **258**, 1–171.

Hestenes, David (1966). *Space-Time Algebra*, Gordon and Breach, New York.

Kähler, Erich (1960). Innerer und äusere Differentialkalkül, *Abhandlunden der Deutschen Akademie der Wissenschaften zu Berlin*, **1960**(4), 1–32.

Kähler, Erich (1961). Die Dirac-Gleichung, *Abhandlunden der Deutschen Akademie der Wissenschaften zu Berlin*, **1961**(1), 1–38.

Kähler, Erich (1962). Der innere Differentialkalkül, *Rendiconti di Matematica e Delle sue Applicazioni*, **21**, 425–523.

Kopczynski, Wojciech (1982). Problems with metric-teleparallel theories of gravitation, *Journal of Physics A*, **15**, 493–506.

Mielke, Eckehard W. (1992). Ashtekar complex variables in general relativity and its teleparallelism equivalent, *Annals of Physics*, **219**, 78–108.

Mielke, Eckehard W. (1999). Anomaly-free solution of Ashtekar constraints for the teleparallelism equivalent of gravity, *Physics Letters A*, **251**, 349–353.

Müller-Hoissen, Fockert, and J. Nitsch (1985). On the tetrad theory of gravity, *General Relativity and Gravitation*, **17**(8), 747–760.

Muraskin, Murray (1995). *Mathematical Aesthetic Principles, Nonintegrable Systems*, World Scientific, Singapore.

Pandres, Dave (1998). Gravitational and electroweak interaction, *International Journal of Theoretical Physics*, **37**, 827–839.

Pandres, Dave (1999). Gravitational and electroweak unification, *International Journal of Theoretical Physics*, **38**, 1783–1805.

Riemann, Bernhard (1854). Über die Hypothesen, welche der Geometrie zu Grunde liegen, Habilitationschrift, Göttingen, in: *Collected Works of Bernhard Riemann*.

Riemann, Bernhard (1861). Comentatio Mathematica, in: *Collected Works of Bernhard Riemann*.

Sakharov, Andrei (1968). Vacuum quantum fluctuations in curved space and the theory of gravitation, *Soviet Physics—Doklady*, **12**(11), 1040–1041.

Schmeikal, Bernd (1996). The generative process of space-time and strong interaction quantum numbers of orientation, in: Rafał Ablamowicz, Pertti Lounesto, and Joseph M. Parra, Editors, *Clifford Algebras with Numeric and Symbolic Computations*, Birkhäuser, Boston, pp. 83–100.

Vacaru, Sergiu I. (1997a). Locally anisotropic gravity and strings, *Annals of Physics*, **256**, 39–61.

Vacaru, Sergiu I. (1997b). Superstrings in higher order extensions of Finsler superspaces, *Nuclear Physics B*, **494**, 590–656.

Vargas, Jose Gabriel, and Douglas Graham Torr (1993). Finslerian structures: The Cartan–Clifton method of the moving frame, *Journal of Mathematical Physics*, **34**(10), 4898–4913.

Vargas, Jose Gabriel, and Douglas Graham Torr (1996). Elementary geometries underlying the theory of Euclidean connections on Finsler metrics of Lorentzian signature, *Contemporary Mathematics*, **196**, 301–310.

Vargas, Jose Gabriel, and Douglas Graham Torr (1997a). The emergence of a Kaluża–Klein microgeometry from the invariants of optimally Euclidean Lorentzian connections, *Foundations of Physics*, **27**, 533–558.

Vargas, Jose Gabriel, and Douglas Graham Torr (1997b). The construction of teleparallel Finsler connections and the emergence of an alternative concept of metric compatibility, *Foundations of Physics*, **27**, 825–843.

Vargas, José Gabriel, and Douglas Graham Torr (1998). Teleparallel Kähler calculus for space-time, *Foundations of Physics*, **28**, 931–958.

Vargas, José Gabriel, and Douglas Graham Torr (1999a). The Cartan–Einstein unification with teleparallelism and the discrepant measurements of Newton's gravitational constant, *Foundations of Physics*, **29**, 145–200.

Vargas, José Gabriel, and Douglas Graham Torr (1999b). The theory of acceleration within its context of differential invariants: The root of the problem with cosmological constants?, *Foundations of Physics*, **29**, 1543–1580.

Vargas, José Gabriel, and Douglas Graham Torr (2000). Clifford-valued Clifforms: a geometric language for Dirac equations, in: Rafał Ablamowicz and Bertfried Fauser, Editors, *Clifford Algebras and Their Applications in Mathematical Physics, Volume 1, Algebra and Physics*, Birkhäuser, Boston, pp. 135–154.

Wu, Tai Sung, and Chen Ning Yang (1975). Concept of nonintegrable phase factor and global formulation of gauge fields, *Physical Review D*, **12**, 3845–3857.